A recursive formula for the evaluation of ⟨ψ|$T_{\mu\nu}$(x)|ψ⟩ and its application in the semiclassical theory of gravity

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A recursive formula for the evaluation of $\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle$ and its application in the semiclassical theory of gravity

S Randjbar-Daemi

International Centre for Theoretical Physics, Trieste, Italy

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Abstract. For a quantum field coupled to a classical background $g_{\mu\nu}$-field we propose a recursive technique which relates the diagonal matrix element $\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle$ to its value at $t=-\infty$. We then employ the lowest non-trivial order to renormalise the semiclassical theory of gravity. The existence of two important classes of solutions of the linearised theory is briefly discussed.

1. Introduction

In recent years there has been a considerable amount of interest in techniques of regularisation of the off-diagonal matrix elements $\langle\text{out}, 0|\hat{T}_{\mu\nu}(x)|0, \text{in}\rangle$ where $\hat{T}_{\mu\nu}$ is the stress-tensor operator of quantised fields propagating in a fixed background space–time with the metric tensor $g_{\mu\nu}$ (for a review see Birrell and Davies 1982).

In § 2 of this paper we would like to carry out a similar investigation, namely we will give a recursive formula to evaluate the diagonal matrix elements $\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle$ associated with a Hermitian scalar field operator $\hat{\phi}$ propagating in a fixed classical gravitational field $g_{\mu\nu}$. These matrix elements, in addition to being interesting in their own right, also arise in a natural manner in the semiclassical theory of gravity, where one couples a classical gravitational field $g_{\mu\nu}$ to the quantised matter through the action integral

$$W[|\psi\rangle, g_{\mu\nu}] = W_{\mathrm{E}}[g_{\mu\nu}] + W_{\mathrm{Sch}}[g_{\mu\nu}, |\psi\rangle]. \tag{1}$$

Here $W_{\mathrm{E}} = (16\kappa^2)^{-1}\int \mathrm{d}^4x\,\sqrt{-g}\,R$ and W_{Sch} is the quantum matter action written in the Schrödinger picture (Kibble and Randjbar-Daemi 1980). The equations of motion derived from equation (1) may be transformed into the Heisenberg picture, in which they read

$$[(-g)^{-1/2}\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu) - \mu^2]\hat{\phi}(x) = 0, \tag{2a}$$

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi\kappa^2\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle, \tag{2b}$$

where $|\psi\rangle$ is a normalisable Heisenberg state vector and κ^2 is Newton's constant.

As in any quantum field theory one can only hope to obtain perturbative solutions to equation (2). For this purpose it is essential to have a technique of evaluating $\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle$ in a step-by-step fashion, for instance, from an initial flat space matrix element $\langle\psi|\hat{T}^0_{\mu\nu}|\psi\rangle$. This is the problem which we solve in § 2. In § 3 we specialise our general formula to the lowest non-trivial order. Then for the choice $|\psi\rangle = |0, \text{in}\rangle$ we apply the dimensional regularisation technique to identify the divergent parts. In

§ 4 we construct a renormalisable action for the semiclassical theory of gravity from which we derive the final form of the linearised renormalised Einstein field equations. We also derive the 'β function' equations for all the parameters of the model. At the end of this section we argue that the semiclassical gravitational equation (2) may be employed to resolve the well known finite renormalisation ambiguities. In § 5 we first summarise the results of the previous sections and then we go on to discuss, though very briefly, the structure of two important classes of solutions of the equations of motion. These are the gravitational wave solution propagating with the velocity of light and the solutions which represent small perturbations of the flat space–time and grow exponentially in time. The existence of this latter class of solutions can be an indication of instability of the state $(\eta_{\mu\nu}, |0, \text{in}\rangle)$.

Notations and conventions

Units: $\hbar = c = 1$, sgn $g_{\mu\nu} = (-, +, +, +)$, the sign ':=' means definition. The Ricci tensor: $R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \ldots$. We also absorb the factors of 2π by putting a bar on d's and δ's, i.e. $\text{đ}^n k := (2\pi)^{-n}\, \text{d}^n k$ and $\bar{\delta}^n(k) := (2\pi)^n \delta^n(k)$.

2. Calculation of $\langle\psi|T_{\mu\nu}(x)|\psi\rangle$

In this section we will assume that $g_{\mu\nu}$ is a fixed asymptotically flat background metric. We also choose a class of harmonic coordinate systems distinguished by the condition

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}) = 0, \qquad \nu = 0, 1, 2, 3. \tag{3}$$

Then it is easily seen that the two-point function $\Delta_\psi(x, x')$ defined by

$$\Delta_\psi(x, x') := \text{i}\langle\psi|T(\hat{\phi}(x)\hat{\phi}(x'))|\psi\rangle \tag{4}$$

satisfies the equation

$$[g^{\mu\nu}(x)(\partial/\partial x^\mu)(\partial/\partial x^\nu) - \mu^2]\Delta_\psi(x, x') = -\delta^4(x, x')(-g(x))^{-1/2}. \tag{5}$$

Clearly as $x \to x'$ the function Δ_ψ becomes undefined. Therefore before taking this limit we have to regularise Δ_ψ. We will define $\langle\psi|\phi^2(x)|\psi\rangle^{\text{reg}}$ and $\langle\psi|\partial_\mu\phi(x)\partial_\nu\phi(x)|\psi\rangle^{\text{reg}}$ via

$$\langle\psi|\hat{\phi}^2(x)|\psi\rangle^{\text{reg}} := [-\text{i}\Delta^{\text{reg}}_\psi(x, x')]_{x=x'}, \tag{6a}$$

$$\langle\psi|\partial_\mu\hat{\phi}(x)\partial_\nu\hat{\phi}(x')|\psi\rangle^{\text{reg}} := [-\text{i}(\partial/\partial x^\mu)(\partial/\partial x'^\nu)\Delta^{\text{reg}}_\psi(x, x')]_{x=x'}. \tag{6b}$$

Having obtained these quantities, we insert them into

$$\begin{aligned}\langle\psi|T_{\mu\nu}(x)|\psi\rangle^{\text{reg}} &:= -\langle\psi|\partial_\mu\hat{\phi}(x)\partial_\nu\hat{\phi}(x)|\psi\rangle^{\text{reg}} + \tfrac{1}{2}g_{\mu\nu}(x)\\ &\quad\times\{g^{\lambda\sigma}(x)\langle\psi|\partial_\lambda\hat{\phi}(x)\partial_\sigma\hat{\phi}(x)|\psi\rangle^{\text{reg}} + \mu^2\langle\psi|\hat{\phi}^2(x)|\psi\rangle^{\text{reg}}\}.\end{aligned} \tag{7}$$

Thus the problem of finding a perturbative scheme of calculating $\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle^{\text{reg}}$ essentially reduces to that of $\Delta_\psi(x, x')$. This can be achieved by solving equation (5) under suitable boundary conditions. We will adopt the following initial conditions: as t and $t' \to -\infty$ then $\Delta_\psi(x, x') \to \Delta^0_\psi(x, x')$ where

$$\Delta^0_\psi(x, x') := \text{i}\langle\psi|T(\hat{\phi}_0(x)\hat{\phi}_0(x'))|\psi\rangle. \tag{8}$$

$\hat{\phi}_0$ is a free Klein–Gordon field, i.e.

$$(\partial^2-\mu^2)\hat{\phi}_0=0. \tag{9}$$

These conditions are compatible with the assumption that initially the space–time is flat. With this initial condition, a convenient way of obtaining a step-by-step solution of equation (5) is to introduce new fields $h^{\mu\nu}$ and h through

$$g^{\mu\nu}(x):=\eta^{\mu\nu}-h^{\mu\nu}(x), \tag{10a}$$

$$(-g(x))^{1/2}:=1+\tfrac{1}{2}h(x), \tag{10b}$$

where

$$\eta^{\mu\nu}=\mathrm{diag}\,(-1,+1,+1,+1).$$

These equations must be interpreted as simple field redefinitions. They do not mean any linearisation. The function $h(x)$ can of course be expressed in terms of $h^{\mu\nu}$, but for the time being we do not need it. We only mention that in the linearised theory $h=\eta_{\mu\nu}h^{\mu\nu}$.

Upon insertion of equation (10) into equation (5), we may write it in an integral form which incorporates the specified initial conditions, i.e.

$$\Delta_\psi(x,x')=\Delta_\psi^0(x,x')-\int d^4y\;\Delta^{\mathrm{R}}(x-y)H(y)\Delta_\psi(y,x'), \tag{11}$$

where $\Delta^{\mathrm{R}}(x-y)$ is the retarded propagator of the operator $(\partial^2-\mu^2)$ and the operator H is defined by

$$H(y):=-\tfrac{1}{2}h(y)\left(\eta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}-\mu^2\right)+(1+\tfrac{1}{2}h(y))h^{\mu\nu}(y)\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}. \tag{12}$$

Although $\Delta_\psi(x,x')$ of equation (11) satisfies equation (5) with respect to its x variable, it does not do so, however, with respect to its x' variable. The reason for this is that as it stands the second term in equation (11) is not manifestly symmetric with respect to the interchange of x and x'. If we simply symmetrise it then it will not satisfy equation (5) with respect to any of its variables. To circumvent this obstacle we first symmetrise the RHS of equation (11) and then add a compensating term to be determined consistently. Thus

$$\begin{aligned}\Delta_\psi(x,x')=\Delta_\psi^0(x,x')-\int \mathrm{d}^4y\;[&\Delta^{\mathrm{R}}(x-y)H(y)\Delta_\psi(y,x')\\ &+\Delta^{\mathrm{R}}(x'-y)H(y)\Delta_\psi(y,x)+\Delta^{\mathrm{R}}(x-y)F_\psi(y)\Delta^{\mathrm{R}}(x'-y)].\end{aligned} \tag{13}$$

The compensating function F_ψ must be chosen in such a way that $\Delta_\psi(x,x')$ meets all of our requirements. Before going any further we rewrite equation (13) in a slightly different form. To this end, first we substitute equation (10) into equation (3) to obtain

$$\partial_\mu H^{\mu\nu}=0 \tag{14a}$$

where

$$H^{\mu\nu}:=\tfrac{1}{2}\eta^{\mu\nu}h-h^{\mu\nu}-\tfrac{1}{2}hh^{\mu\nu}. \tag{14b}$$

Next we substitute equation (12) into equation (13) and make use of (14). After some integrations by parts we obtain

$$\Delta_\psi(x, x') = \Delta^0_\psi(x, x') - \int d^4y \Big\{ \tfrac{1}{2}\mu^2[\Delta^R(x-y)h(y)\Delta_\psi(y, x') + x \leftrightarrow x'] + \left[\frac{\partial}{\partial y^\mu}\Delta^R(x-y)H^{\mu\nu}(y)\frac{\partial}{\partial y^\nu}\Delta_\psi(y, x') + x \leftrightarrow x'\right] + \Delta^R(x-y)F_\psi(y)\Delta^R(x'-y)\Big\}. \tag{15}$$

Now we are ready to specify the function F_ψ by imposing the condition that $\Delta_\psi(x, x')$ satisfies equation (5) in both of its variables x and x'. Thus if we operate on (15) by $(\eta^{\mu\nu}(\partial/\partial x^\mu)(\partial/\partial x^\nu) - \mu^2)$, say, we then obtain the following consistency equation for F_ψ:

$$\Delta^R(x-x')F_\psi(x) = \int d^4y \Big(\tfrac{1}{2}\mu^2\Delta^R(x'-y)h(y)(\partial_x^2 - \mu^2)\Delta_\psi(y, x) + \frac{\partial}{\partial y^\mu}\Delta^R(x'-y)H^{\mu\nu}(y)\frac{\partial}{\partial y^\nu}(\partial_x^2 - \mu^2)\Delta_\psi(y, x)\Big). \tag{16}$$

Hence the recursive solution for $\Delta_\psi(x, x')$ in a given $g_{\mu\nu}$ field must be evaluated in the following order.

We start with the zeroth-order solution $\Delta^0_\psi(x, x')$ of equation (15) and substitute it into equation (16) to obtain the first-order solution for F_ψ. This together with Δ^0_ψ for Δ_ψ must be inserted into the RHS of equation (15) to yield the first-order solution for Δ_ψ, which in turn must be replaced in equation (16) to produce the second-order solution for F_ψ and so on. These steps may be repeated an arbitrary number of times.

It is worth mentioning that here we are dealing with the series expansion for Δ_ψ in the full background field $g_{\mu\nu}$ (or rather $h^{\mu\nu}$). At each stage of this expansion we may in turn expand $h^{\mu\nu}$ (and h) in a power series of some parameter, the Newtonian constant say. In a perturbative treatment of equation (2) these two expansions must of course go hand in hand.

Now, as an example (and for application in the next section) let us choose $|\psi\rangle = |0, \text{in}\rangle$ and calculate $\Delta_\psi(x, x')$ up to the lowest non-trivial order. For this choice of $|\psi\rangle$ the lowest-order solution is of course trivial. It is the free Feynman propagator $\Delta_F(x, x')$. If we substitute this into equation (16) we obtain (we omit the subscript $|\psi\rangle$)

$$\Delta^R(x-x')F_{(1)}(x) = \frac{-\mu^2}{2}\Delta^R(x'-x)h(x) + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}(\Delta^R(x'-x)H^{\mu\nu}(x)).$$

To obtain the next-order solution this expression for $F_{(1)}$ as well as Δ_F for Δ^0_ψ must be inserted into equation (15). In doing this we also make use of the identity

$$\Delta^R(x-y) = \Delta^F(x-y) - \Delta^{(-)}(x-y), \tag{17}$$

where $\Delta^{(-)}$ is the negative frequency function defined by

$$\Delta^{(-)}(x-y) = i\int d^4p\, \theta(p^0)\delta(p^2+\mu^2)\, e^{-ip(x-y)} \tag{18}$$

where $\theta(p^0)$ is the step function.

With these substitutions we obtain the following first-order solution $\Delta_{(1)}$:

$$\begin{aligned}\Delta_{(1)}(x,x') = \Delta_F(x-x') - \int d^4y \bigg(\frac{\mu^2}{2}[\Delta_F(x-y)h(y)\Delta_F(x'-y)\\ - \Delta^{(-)}(x-y)h(y)\Delta^{(-)}(x'-y)]\\ + \frac{\partial}{\partial y^\mu}\Delta_F(x-y)H^{\mu\nu}(y)\frac{\partial}{\partial y^\mu}\Delta_F(x'-y)\\ - \frac{\partial}{\partial y^\mu}\Delta^{(-)}(x-y)H^{\mu\nu}(y)\frac{\partial}{\partial y^\nu}\Delta^{(-)}(x'-y)\bigg).\end{aligned} \tag{19}$$

For the application in the next sections it is convenient to introduce a new notation Δ_h,

$$\begin{aligned}\Delta_h(x,x') := -\int d^4y\bigg(\frac{\mu^2}{2}\Delta^{(-)}(x-y)h(y)\Delta^{(-)}(x'-y)\\ + \frac{\partial}{\partial y^\mu}\Delta^{(-)}(x-y)H^{\mu\nu}(y)\frac{\partial}{\partial y^\nu}\Delta^{(-)}(x'-y)\bigg).\end{aligned} \tag{20}$$

Then equation (19) may be written as

$$\Delta_{(1)}(x,x') = \Delta_{inh}(x,x') + \Delta_h(x,x'), \tag{21}$$

where $\Delta_{inh}(x,x')$ is defined in a self-evident way.

The subscript on Δ_h is to remind us that it satisfies the homogenous Klein–Gordon equation, i.e.

$$(\partial^2 - \mu^2)\Delta_h(x,x') = 0.$$

$\Delta_{inh}(x,x')$ on the other hand will satisfy an inhomogenous equation. Hence the Δ_h term in equation (21) has the effect of ensuring the correct boundary condition satisfied by $\Delta_{(1)}$. (It is obvious from equation (19) that $\Delta_{(1)}$ satisfies the prescribed initial condition.)

To go to higher-order terms we must substitute equation (21) into equation (16) to get the second-order F and then replace the result in equation (15) to calculate the second-order $\Delta_{|0,in\rangle}(x,x')$. This procedure may be repeated up to any desired order.

3. The linearised theory

Let us assume that all the components of $h^{\mu\nu}$ satisfy the inequalities $|h^{\mu\nu}| \ll 1$. Then neglecting the second and higher powers of $h^{\mu\nu}$ we obtain

$$h = \eta_{\mu\nu}h^{\mu\nu}, \tag{22a}$$

$$H^{\mu\nu} = \tfrac{1}{2}\eta^{\mu\nu}h - h^{\mu\nu}. \tag{22b}$$

We shalll also assume that $|\psi\rangle = |0, \text{in}\rangle$ and then substitute equation (22) into (21) and regularise $\Delta_{(1)}$.

It is interesting to note that the contribution of $\Delta_h(x,x')$ in equation (21) is finite as $x \to x'$. To see this, we insert the Fourier transforms of different Δ-functions into the linearised version of equation (20) to obtain

$$\Delta_{\rm h}(x, x') = \int d^4q\, \tilde{h}^{\lambda\sigma}(q) \int d^4p\, \exp[-ixp + ix'(p+q)] \times (\theta(p^0)\delta(p^2+\mu^2)\theta(-p^0-q^0)\delta(q^2+2qp)p_\lambda p_\sigma). \tag{23}$$

Since, as $x \to x'$, the range of the p integral is confined to the finite intersection of the two overlapping cones defined by

$$\theta(p^0)\delta(p^2+\mu^2) \qquad \text{and} \qquad \theta(-p^0-q^0)\delta(q^2+2qp),$$

the integral can never diverge. In fact, we have explicitly calculated this integral in the appendix. Its contribution to $\langle 0, \text{in}\,|\hat{T}_{\mu\nu}(x)|0, \text{in}\rangle^{\text{reg}}$ is given by

$$\begin{aligned}
&\langle 0, \text{in}\,|\hat{T}_{\mu\nu}(x)|0, \text{in}\rangle_{\rm h} \\
&\quad := -i \lim_{x\to x'} \left\{\left[-\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} + \frac{1}{2}\eta_{\mu\nu}\left(\eta^{\lambda\sigma}\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial x^\sigma} + \mu^2\right)\right]\Delta_h(x, x')\right\} \\
&\quad = -\tfrac{1}{15}i \int d^4q\, e^{iqx} I(q)[-(q_\mu q_\nu/q^2)(\mu^2 + \tfrac{1}{4}q^2)(-\tfrac{3}{2}\mu^2 + \tfrac{7}{8}q^2)\tilde{h}(q) \\
&\quad\quad - \tfrac{15}{32}q_\mu q_\nu q^2 \tilde{h}(q) + 2(\mu^2 + \tfrac{1}{4}q^2)^2 \tilde{h}_{\mu\nu}(q)] \\
&\quad\quad - \tfrac{1}{10}i\,\eta_{\mu\nu} \int d^4q\, e^{iqx}[I(q)\tilde{h}(q)(\tfrac{1}{3}\mu^4 + \tfrac{1}{6}\mu^2 q^2 + \tfrac{1}{8}q^4)],
\end{aligned} \tag{24}$$

where

$$I(q) := (4\pi)^{-1}(\tfrac{1}{4} + \mu^2/q^2)^{1/2}\theta(-q^2 - 4\mu^2)\theta(-q^0). \tag{25}$$

In contrast to $\Delta_{\rm h}$, the contribution of $\Delta_{\rm inh}$ to $\langle 0, \text{in}\,|\hat{T}_{\mu\nu}(x)|0, \text{in}\rangle^{\text{reg}}$ becomes infinite as soon as $x \to x'$. Therefore we have to adopt some regularisation scheme. Here we employ the dimensional regularisation scheme ('t Hooft and Veltman 1972). To this end, first we substitute equation (22) into equations (19)–(21) to obtain

$$\begin{aligned}
\Delta_{\rm inh}(x, x') = \Delta_{\rm F}(x-x') - \int d^4y \Big(&\frac{\mu^2}{2}\Delta_{\rm F}(x-y)h(y)\Delta_{\rm F}(x'-y) \\
&+ \tfrac{1}{2}h(y)\eta^{\mu\nu}\frac{\partial}{\partial y^\mu}\Delta_{\rm F}(x-y)\frac{\partial}{\partial y^\nu}\Delta_{\rm F}(x'-y) \\
&- h^{\mu\nu}(y)\frac{\partial}{\partial y^\mu}\Delta_{\rm F}(x-y)\frac{\partial}{\partial y^\nu}\Delta_{\rm F}(x'-y)\Big).
\end{aligned}$$

Then we replace the Fourier transforms of $\Delta_{\rm F}$ and $h_{\mu\nu}$ and adopt the dimensional regularisation by first analytically continuuing into four-dimensional Eculidean space and then letting the dimension of space become an arbitrary complex number n. After a relatively long but straightforward calculation we are led to the following results:

$$\langle 0, \text{in}\,|\hat{T}_{\mu\nu}(x)|0, \text{in}\rangle^{\text{reg}}_{\text{inh}} = \langle \hat{T}_{\mu\nu}(x)\rangle^{\text{div}}_{\text{inh}} + \langle \hat{T}_{\mu\nu}(x)\rangle^{\text{f}}_{\text{inh}}, \tag{26a}$$

$$\begin{aligned}
\langle \hat{T}_{\mu\nu}(x)\rangle^{\text{div}}_{\text{inh}} = \frac{\Gamma(\varepsilon)}{(4\pi)^{n/2}m^{2\varepsilon}}\Big(&\frac{1}{4}\frac{\mu^4}{\mu_{\rm R}^{2\varepsilon}}g_{\mu\nu}(x) - \tfrac{1}{12}\mu^2\partial^2 h_{\mu\nu}(x) \\
&+ \tfrac{1}{24}\mu^2\delta_{\mu\nu}\partial^2 h(x) + \tfrac{1}{120}\partial^4 h_{\mu\nu}(x) - \tfrac{1}{60}\partial_\mu\partial_\nu\partial^2 h(x) + \tfrac{1}{80}\delta_{\mu\nu}\partial^4 h(x)\Big),
\end{aligned} \tag{26b}$$

$$\langle T_{\mu\nu}(x)\rangle^{\mathrm{f}}_{\mathrm{inh}}=\frac{\varepsilon\Gamma(\varepsilon)}{(4\pi)^{n/2}m^{2\varepsilon}}\left(\Phi_{\mu\nu}(x)+\frac{3}{8}\frac{\mu^4}{\mu_{\mathrm{R}}^{2\varepsilon}}g_{\mu\nu}(x)\right). \tag{27a}$$

Here m is an arbitrary unit of mass such that $\mu=m\mu_{\mathrm{R}}$ with μ_{R} dimensionless and $\varepsilon=2-n/2$. The function $\Phi_{\mu\nu}(x)$, which remains finite as $\varepsilon\to 0$, is defined by

$$\begin{aligned}\Phi_{\mu\nu}(x):=&\frac{\mu^4}{4\mu_{\mathrm{R}}^{2\varepsilon}}h_{\mu\nu}(x)\log\mu_{\mathrm{R}}^2-\frac{\mu^4}{80\mu_{\mathrm{R}}^{2\varepsilon}}h(x)\log\mu_{\mathrm{R}}^2-\tfrac{1}{8}\mu^2(\partial^2h_{\mu\nu}(x)+\tfrac{3}{2}\delta_{\mu\nu}\partial^2h(x))\\&+\tfrac{1}{80}\partial^4h_{\mu\nu}(x),+\tfrac{1}{6}\partial_\mu\partial_\nu\partial^2h(x)-\tfrac{23}{480}\delta_{\mu\nu}\partial^4h(x)\\&+\mathrm{i}m^4\int_0^1\mathrm{d}\alpha\int\mathrm{d}^nq\ \mathrm{e}^{\mathrm{i}qx}\log\left(\mu_{\mathrm{R}}^2+\frac{q^2}{m^2}(\alpha-\alpha^2)\right)\\&\times\left\{\frac{\tilde{h}_{\mu\nu}(q)}{4}\left(\mu_{\mathrm{R}}^2+\frac{q^2}{m^2}(\alpha-\alpha^2)\right)^2\right.\\&+\tilde{h}(q)\left[\frac{q^2q_\lambda q_\sigma}{m^4}\left(\frac{\alpha^4}{2}-\alpha^3+\frac{\alpha^2}{2}\right)+\frac{q^4}{m^4}\left(\frac{-3}{8}\alpha^4+\frac{3}{4}\alpha^3-\frac{3}{8}\alpha^2\right)\right.\\&\left.\left.+\frac{q^2}{m^2}\mu_{\mathrm{R}}^2\left(\frac{-\alpha^2}{4}+\frac{\alpha}{4}\right)-\frac{\mu_{\mathrm{R}}^2}{8}\right]\right\}.\end{aligned}\tag{27b}$$

When the regularisation is lifted (i.e. $\varepsilon\to 0$) we have to continue equation (27b) back into the four-dimensional Minkowski space by letting $\mathrm{i}\,\mathrm{d}^nq\to-\mathrm{d}^4q$. The divergent part of $\langle\chi|T_{\mu\nu}(x)|\psi\rangle$ is of course independent of the choice of the states $|\chi\rangle$ and $|\psi\rangle$ (DeWitt 1975). In fact, it is easy to see that $\frac{1}{2}\sqrt{-g}\langle\hat{T}_{\mu\nu}\rangle^{\mathrm{div}}_{\mathrm{inh}}$ may be obtained from the linearisation of $(\delta/\delta g^{\mu\nu})\Delta\mathscr{L}$ where

$$\Delta\mathscr{L}=[g^{1/2}/8\pi^2(n-4)](\tfrac{1}{4}\mu^4-\tfrac{1}{12}\mu^2R+\tfrac{1}{120}R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{240}R^2). \tag{28}$$

This is, of course, a well known result ('t Hooft and Veltman 1974). The finite part of $\langle\chi|T_{\mu\nu}|\psi\rangle$ does on the other hand depend on the choice of the states $|\psi\rangle$ and $|\chi\rangle$. In our problem, for which $|\chi\rangle=|\psi\rangle=|0,\mathrm{in}\rangle$, we have

$$\langle\hat{T}_{\mu\nu}(x)\rangle^{\mathrm{f}}:=\langle\hat{T}_{\mu\nu}(x)\rangle^{\mathrm{f}}_{\mathrm{inh}}+\langle\hat{T}_{\mu\nu}(x)\rangle_{\mathrm{h}}$$

where $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{h}}$ and $\langle\hat{T}_{\mu\nu}\rangle^{\mathrm{f}}_{\mathrm{inh}}$ are given by equations (25) and (27a, b) respectively.

It is well known that the identification of $\langle\hat{T}_{\mu\nu}\rangle^{\mathrm{f}}$ is not unique (Birrell and Davies 1982). It has been argued elsewhere that the inclusion of the dynamics of the background field may be employed to resolve the ambiguity (Kay *et al* 1980). We do this in the next section.

4. Renormalisation

Consider the action integral

$$W[g_{\mu\nu},|\psi\rangle]=(4\pi)^{-n/2}\int\mathrm{d}^nx\ \sqrt{g}(\Lambda_{\mathrm{B}}+g_{\mathrm{B}}R+\lambda_{\mathrm{B}}R^2+\gamma_{\mathrm{B}}R_{\mu\nu}R^{\mu\nu})+W_{\mathrm{Sch}}[g_{\mu\nu},|\psi\rangle], \tag{29}$$

where the integration is over an unphysical n-dimensional Riemanian manifold with the positive definite metric tensor $g_{\mu\nu}$. In the system of natural units $\hbar=c=1$, the bare parameters Λ_{B} and g_{B} have the dimensionality of $[\mathrm{length}]^{-n}$ and $[\mathrm{length}]^{-n+2}$ respectively, while both λ_{B} and γ_{B} have the dimension $[\mathrm{length}]^{4-n}$. W_{Sch} is the same as explained in the introduction. If we choose the class of coordinate systems specified by equation (3) to linearise $\delta W(g_{\mu\nu})/\delta g_{\mu\nu}=0$ by setting $g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}$, we then obtain

$$(4\pi)^{-n/2}[-\tfrac{1}{2}\Lambda_B\, g_{\mu\nu}(x)+g_B(-\tfrac{1}{2}\partial^2 h_{\mu\nu}(x)+\tfrac{1}{4}\delta_{\mu\nu}\partial^2 h(x))+(\lambda_B+\tfrac{1}{2}\gamma_B)\partial_\mu\partial_\nu\partial^2 h(x)$$
$$-(\lambda_B+\tfrac{1}{4}\gamma_B)\delta_{\mu\nu}\partial^4 h(x)-\tfrac{1}{2}\gamma_B\partial^4 h_{\mu\nu}(x)]+\tfrac{1}{2}\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle^{\mathrm{reg}}=0.$$

For the choice of $|\psi\rangle=|0,\mathrm{in}\rangle$ we can substitute

$$\langle 0,\mathrm{in}|\hat{T}_{\mu\nu}(x)|0,\mathrm{in}\rangle^{\mathrm{reg}}=\langle\hat{T}_{\mu\nu}(x)\rangle^{\mathrm{div}}_{\mathrm{inh}}+\langle\hat{T}_{\mu\nu}(x)\rangle^{\mathrm{f}}_{\mathrm{inh}}+\langle\hat{T}_{\mu\nu}(x)\rangle_{\mathrm{h}}.$$

If we also make use of equations (26) and (27), we then obtain

$$-m^4\Lambda_R g_{\mu\nu}(x)+m^2 g_R G_{\mu\nu}(x)+(\lambda_R+\tfrac{1}{2}\gamma_R)\partial_\mu\partial_\nu\partial^2 h(x)-(\lambda_R+\tfrac{1}{4}\gamma_R)\delta_{\mu\nu}\partial^4 h(x)$$
$$-\tfrac{1}{2}\gamma_R\partial^4 h_{\mu\nu}(x)+\tfrac{1}{2}\Phi^{(\varepsilon\to 0)}_{\mu\nu}(x)+8\pi\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{h}}=0 \qquad (30)$$

where the ε-independent parameters Λ_R, g_R, λ_R and γ_R are defined by

$$\Lambda_B := m^{4-2\varepsilon}[\Lambda_R+(\mu_R^{4-2\varepsilon}/4\varepsilon)(1+\tfrac{3}{2}\varepsilon)], \qquad (31a)$$

$$g_R := m^{2-2\varepsilon}(g_R-\mu_R^2/12\varepsilon), \qquad (31b)$$

$$\gamma_B := m^{-2\varepsilon}(\gamma_R+1/120\varepsilon), \qquad (31c)$$

$$\lambda_B := m^{-2\varepsilon}(\lambda_R+1/240\varepsilon), \qquad (31d)$$

(we have substituted $\Gamma(\varepsilon)$ by $1/\varepsilon$).

Equation (30) has been written in the four-dimensional Minkowski space. The independence of the bare parameters from the choice of m will make the renormalised parameters depend on m. This dependence is governed by the 'β function' equations which are obtained by differentiating equation $(31a)$–$(31d)$ with respect to m and then letting $\varepsilon\to 0$. To do this we introduce a scale parameter s through $m=m_0\exp(s)$, and we then get

$$\partial\mu_R(s)/\partial s=-\mu_R(s), \qquad (32a)$$

$$\partial\Lambda_R(s)/\partial s=-4\Lambda_R(s), \qquad (32b)$$

$$\partial g_R(s)/\partial s=-2g_R(s)-\tfrac{1}{6}\mu_R^2(2). \qquad (32c)$$

$$\partial\gamma_R(s)/\partial s=\tfrac{1}{60}, \qquad (32d)$$

$$\partial\lambda_R(s)/\partial s=\tfrac{1}{120}. \qquad (32e)$$

These equations can easily be integrated to yield

$$\begin{aligned}&\mu_R(s)=\mu_R(0)\mathrm{e}^{-s}, \qquad \Lambda_R(s)=\Lambda_R(0)\mathrm{e}^{-4s}, \qquad \gamma_R(s)=\tfrac{1}{60}s+\gamma_R(0),\\ &g_R(s)=g_R(0)\mathrm{e}^{-2s}+\tfrac{1}{6}\mu_R^2(s)\log(\mu_R(s)/\mu_R(0)), \qquad \lambda_R(s)=\tfrac{1}{120}s+\lambda_R(0).\end{aligned} \qquad (33)$$

The equations $(32a)$–$(32e)$ ensure that the change in the scale of m does not alter our subtraction prescription ('t Hooft 1973). They also guarantee that the renormalised Einstein equation (30) does not depend on s. Thus it is only the parameters $\mu_R(0)$, $\Lambda_R(0)$, $g_R(0)$, $\gamma_R(0)$, $\lambda_R(0)$ and $m(0)$ which can enter equation (30). Regarding equation (30) as an equation which determines the components of $g_{\mu\nu}$, one may impose restrictions on its solution so that the parameters $\Lambda_R(0)$, $g_R(0)$, $\gamma_R(0)$ and $\lambda_R(0)$ are determined. For instance, one may demand that if initially (i.e. at $t=-\infty$) the quantum field is in the ground state $|0,\mathrm{in}\rangle$ and $g_{\mu\nu}=\eta_{\mu\nu}$ then nothing happens, i.e. this state remains as it is. This will imply $\Lambda_R(0)=0$ which fixes the subtraction $(31a)$. Similar conditions may be invoked for the other parameters. This removes the finite renormalisation ambiguity (Kibble 1981).

It is worth mentioning that the renormalised equation (30) is a partial differential equation of fourth order. Although for $\Lambda_R(0)=0$ it admits solutions which grow exponentially in time (Horowitz 1980, Randjbar-Daemi 1980, 1981), this is however not so fatal for a classical theory of gravity. Indeed, recently Logunov *et al* (1979, 1980) have proposed a new theory of gravity which has a partial differential equation of sixth order as its fundamental equation for the gravitation field.

5. Summary and discussion

In the foregoing sections we proposed a technique for the calculation of the diagonal matrix elements $\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle$ of the stress–tensor operator $\hat{T}_{\mu\nu}$ of a massive Hermitian quantum field $\hat{\phi}$ propagating in a background classical field $g_{\mu\nu}$. Then we studied the lowest non-trivial order in detail and after adopting the ε technique we identified the divergent parts of $\langle 0, \text{in}\,|\hat{T}_{\mu\nu}(x)|0, \text{in}\rangle$ and showed that they are obtained from the variation of a counter Lagrangian given by the master formula of 't Hooft and Veltman (1974).

In the absence of a self-interaction term for $\hat{\phi}$ this is the only counter term needed to remove the infinites up to any order. We incorporated this statement in the model by constructing a renormalisable action integral which we argued may also be employed to resolve the finite renormalisation ambiguities.

Having obtained the renormalised linearised Einstein equation (30), one may now proceed with the investigation of the existence and properties of its solutions. Here we will briefly discuss two important types of solutions, both associated with $\Lambda_R(0)=0$.

The first class of solutions consists of those for which the Fourier transform of (30) admits $h_{\mu\nu}(q)\neq 0$ and $q^2=0$. This is easily checked by noting that from equation (25) $I(q^2=0)=0$ so that equation (24) yields $\langle\hat{T}_{\mu\nu}(q^2=0)\rangle_h=0$. Similarly one can see from expansion (27*b*) that $\phi_{\mu\nu}^{\varepsilon\to 0}(q^2=0)=0$. These solutions represent gravitational waves propagating with the velocity of light.

The second class of solutions are those for which q^2 is space-like. These solutions can in principle be an indication of instability of the state $(\eta_{\mu\nu};\ |0, m\rangle)$. We have discussed this point elsewhere (Randjbar-Daemi 1980, 1981). Here we would like only to mention that this result can also be interpreted as a symptom of breakdown of the perturbation theory. In this case it will be reminiscent of the well known 'Landau ghost' in quantum electrodynamics (Bogoliubov and Shirkov 1980).

Finally the last remark on the semiclassical theory of gravity concerns the time evolution of the quantum state. It is well known (Kibble 1978) that if one regards the model of the universe based on equations (1) and (2) as an exact theory in its own right (i.e. not an approximation to a fully quantised theory (Isham 1981)), then the Schrödinger equation for $|\psi(t)\rangle$ will become nonlinear. Thus the superposition principle of states will be violated, and as a consequence of this there will be particle production out of single-particle initial states (Kay *et al* 1980).

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Appendix. Evaluation of $\Delta_h(x, x')$ in equation (23)

From equation (23) as $x \to x'$ we need to evaluate the following integral:

$$I_{\lambda\sigma}(q) = \int \mathrm{d}^4p\{\theta(p^0)\delta(p^2+\mu^2)\theta(-p^0-q^0)\delta(q^2+2qp)p_\lambda p_\sigma\}. \tag{A1}$$

Because of the Lorentz invariance, the most general form of $I_{\lambda\sigma}(q)$ is

$$I_{\lambda\sigma}(q) = (q_\lambda q_\sigma/q^2)a_1(q) + \eta_{\lambda\sigma}a_2(q) \tag{A2}$$

where $a_1(q)$ and $a_2(q)$ are invariant functions of q. If we contract (A1) and (A2) with $\eta^{\lambda\sigma}$ and $q^\lambda q^\sigma/q^2$ successively, we obtain

$$a_1(q) + 4a_2(q) = -\mu^2 I(q) \tag{A3}$$

$$a_1(q) + a_2(q) = \tfrac{1}{4}q^2 I(q) \tag{A4}$$

where

$$I(q) := \int \mathrm{d}^4p(\theta(p^0)\theta(-p^0-q^0)\delta(p^2+\mu^2)\delta\,(q^2+2pq))$$

$$= (4\pi)^{-1}(\tfrac{1}{4}+\mu^2/q^2)^{1/2}\theta(-q^2-4\mu^2)\theta(-q^0).$$

One can easily obtain a_1 and a_2 from equations (A3) and (A4). In calculating $\langle T_{\mu\nu}\rangle_\mathrm{h}$ of equation (24), one needs to evaluate integrals of the form (A1), but with an integrand involving the product of $P_\lambda P_\sigma P_\gamma$ and $P_\lambda P_\sigma P_\gamma P_\mu$. These integrals may be evluated in a similar way.

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